The Vector Space of Convex Curves: How to Mix Shapes

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We present a novel, log-radius profile representation for convex curves and define a new operation for combining the shape features of curves. Unlike the standard, angle profile-based methods, this operation accurately combines the shape features in a visually intuitive manner. This method have implications in shape analysis as well as in investigating how the brain perceives and generates curved shapes and motions.

INTRODUCTION

Understanding how we perceive and understand shape is a central problem in human and computer vision. Plane curves are the simplest forms that have shape: we can easily recognize objects in cartoons or outlines of images. Here, we introduce a novel way to represent and combine the shape of plane curves that has the twin virtues of mathematical elegance and intuitive simplicity.

ANGLE PROFILE REPRESENTATION

A plane curve is formally described as a continuous mapping from a closed interval of real numbers to a 2-D Euclidean plane, $\Gamma : \mathbb{I} \to \mathbb{R}^2$, *i.e.,* $\Gamma(s) = (x(s), y(s))$, where $s \in \mathbb{I}$ is the 1-D coordinate parameterization along the curve. If the natural, *arc-length* coordinate is used (such that $ds = \sqrt{dx^2 + dy^2}$), the derivative of the curve with respect to s yields a unit-length velocity vector: $\|\Gamma'(s)\| = 1$. This implies that the curve shape can be fully characterized by the velocity vector's orientation along the curve, *i.e.,* the *angle profile* $\{\theta(s)\}$, since the length of the vector does not contain any shape information.

The angle profile representation is widely used for identification and categorization of curve shapes, as well as shape synthesis [1]. In particular, one of the most widely used shape description methods, called *Fourier Descriptors*, analyzes angle profiles of curves as linear combinations of their frequency components [2, 3]. However, such analysis has serious drawbacks, because the operation for combining angle profiles does not properly translate to combination of shape features. (See Appendix A).

Here, we introduce a novel, *log-radius profile* representation that is dual to the angle profile representation. This representation resolves the problems of the angle profile representation, and provides a new operation for combining shape features.

LOG-RADIUS PROFILE REPRESENTATION OF CONVEX CURVES

Consider a subset of plane curves with monotonically increasing angle profiles, called *convex curves*. Such a curve admits an alternative, angle-based parameterization, $\Gamma(\theta)$, since any point on the curve can be uniquely specified by θ. The angle coordinate, like arc-length, is a natural parameterization of a curve, determined uniquely by the curve's geometry. Moreover, it is scale invariant — the coordinate value of a point remains unchanged under scaling operations on the curve.

Differentiating a curve with respect to θ yields a velocity vector whose length is the local radius of curvature:

$$\|\Gamma'(\theta)\| = \frac{ds}{d\theta} \equiv r(\theta) > 0. \tag{1}$$

Then, the shape of a convex curve can be fully characterized by the *radius profile* $\{r(\theta)\}$ along the curve, be-
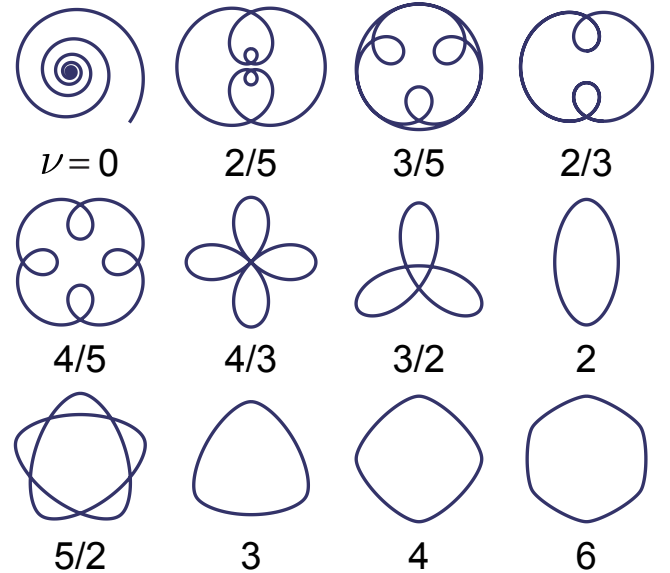

FIG. 1: Examples of elementary shapes shown with their characteristic frequency ν. See https://www.youtube.com/watch?v=waXWOv0YqFE for a movie showing how the shape varies with continuously changing frequency.

cause the orientation of the velocity vector only provides redundant information that is already specified by the angle coordinate.

To summarize, the radius-profile describes the length of the velocity vector as a function of the orientation, whereas the angle-profile describes the orientation of the velocity vector as a function of the arc-length: They provide complementary ways to represent curves.

However, since we will consider scaling operations on curves, it is more convenient to introduce the *log-radius profile* representation, $\{l(\theta)\}$, defined as

$$l(\theta) \equiv \log r(\theta). \tag{2}$$

ELEMENTARY SHAPES

A circle is a simple featureless curve, described by a constant log-radius of curvature. However, interesting shape features are described by fluctuations of the log-radius profiles.

Let us define *elementary shapes* by sinusoidal log-radius profiles:

$$l(\theta) = \epsilon \sin(\nu(\theta - \theta_o)), \tag{3}$$

where ν is the frequency, ϵ is the amplitude and θ_o is the phase shift, which rotates the shape (Fig. 1).

Each elementary shape exhibits a distinctive feature characterized by the frequency ν. For example, the elongated shape of an elliptic curve is characterized by frequency 2, whose log-radius profile oscillates twice per one full rotation of θ, or 2π radians. At larger integer frequencies, the shapes resemble rounded regular polygons. In general, an elementary shape with a rational frequency $\nu = m/n$, where m and n are coprime integers (*i.e.* no common factors) and $m > 1$, has a closed shape of period $\Theta = 2\pi n$, and exhibits m degrees of rotational symmetry. If $m = 1$, then the curve does not close and exhibits a translational symmetry. In the zero frequency limit, the elementary shape approaches a logarithmic spiral: $l(\theta) = \lim_{\nu \to 0} (a/\nu) \sin(\nu\theta) = a\theta$.

The amplitude ϵ modulates the degree of expression of elementary shapes, while preserving their characteristic features (see Fig 2A).

VECTOR SPACE OF CONVEX CURVES

Uniform scaling is the simplest operation on a curve, which scales the overall size of the curve while preserving its shape. In our representation, this corresponds to adding a constant to the log-radius profile.

More generally, we define two operations on convex curves, $\Gamma_i : \mathbb{I} \to \mathbb{R}^2$: scalar multiplication (Fig 2A)

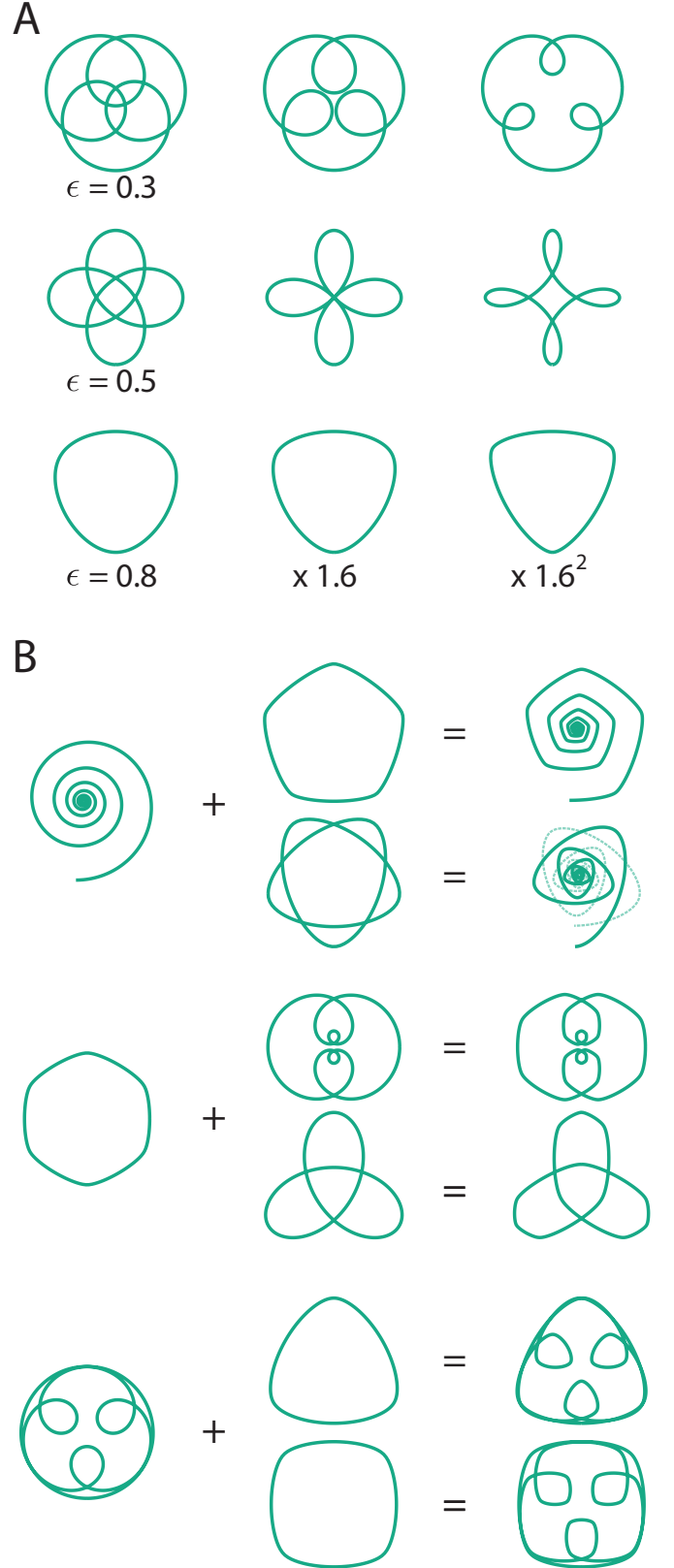$$\Gamma = a \cdot \Gamma_o \longleftrightarrow l(\theta) = a \cdot l_o(\theta), \tag{4}$$

FIG. 2: (A) Scalar multiplication of curves, eq (4). Size is normalized. (B) Addition operation between two curves, eq (5). This operation accurately combines the shape features of curves.

which amplifies the log-radius profile by a scalar factor ($a \in \mathbb{R}, \forall \theta \in \mathbb{I}$), and addition between two curves (Fig 2B)

$$\Gamma = \Gamma_1 + \Gamma_2 \;\longleftrightarrow\; l(\theta) = l_1(\theta) + l_2(\theta). \qquad (5)$$

which adds their log-radius profiles in a pointwise manner ($\forall \theta \in \mathbb{I}$).

The addition operation can be understood as a generalized scaling operation, which uses one curve's log-radius profile as the scale factor for modifying the other curve. It reduces to uniform scaling if one of the curves is a circle. The unit circle, $l(\theta) = 0$, is the identity element of the addition operation.

Remarkably, the addition operation exquisitely merges the shape features of the added curves in a visually intuitive manner (See Fig 2B). For examples, adding a shrinking spiral progressively decreases the length scale of the curve, thereby "spiralizing" its shape, whereas adding an ellipse elongates the curve in one direction and compresses it in the perpendicular direction, thereby "elliptizing" the curve. Thus, adding a spiral and an ellipse produces an elliptic spiral, which combines the shape features of both curves. This accurate combination of shape features owes to the angle coordinate representation, which is invariant under scaling operations.

The scalar multiplication and addition operations define a vector space over the set of convex curves, which is spanned by the basis set of elementary shape curves eq (3); that is, any curve in this space can be represented as a linear combination of elementary curves. Moreover, an inner-product between curves can be defined as,

$$\langle \Gamma_1, \Gamma_2 \rangle \equiv \int_{\theta \in \mathbb{I}} l_1(\theta)\, l_2(\theta)\, d\theta, \qquad (6)$$

which induces a norm $\|\Gamma\| \equiv \sqrt{\langle \Gamma, \Gamma \rangle}$. Thus, convex curves with finite norm ($\|\Gamma\| < \infty$) form a Hilbert space, isomorphic to the space of square-integrable functions, L_2.

DISCUSSION

We presented a novel, log-radius profile representation for describing convex shapes, which is dual to the standard, angle profile-based representation and offers a complementary view of curve shapes. The angle profile representation is closely related to "bending" operations: the simplest curve is a straight line, which can be bent into various curves. In contrast, the log-radius profile representation is closely related to "scaling" operations: the simplest curve is a circle, which can be non-uniformly scaled into other curve shapes.

The log-radius profile representation resolves the aforementioned problems of the angle profile representation (See Appendix A): The elementary curves preserve their characteristic shape features over all range of amplitude, and the addition operation accurately combines the shape features. Therefore, Fourier transform of log-radius profiles indeed properly analyzes the curve shapes into visually meaningful shape features.

Recent applications of this method have revealed surprising details of regularities in kinematics of curved hand movements [4], as well as in speed perception of curved motion [5]. It may also be useful in investigating the shape perception process in vision.

Here, we considered representation of 1-D convex curves in 2-D space. This result can be generalized to higher dimensional, convex surfaces. The angle coordinate then becomes related to normal vector to the surface and inverse Gauss map.

Appendix A: Fourier Descriptors

The Fourier Descriptor method describes the angle profile of a curve as a linear combination of frequency components

$$\theta(s) = s\Theta + \sum_{k=1}^{\infty} a_k \cos(ks\Theta) + b_k \sin(ks\Theta), \qquad (A1)$$

where $s \in [0, 1]$ is the normalized arc-length coordinate and Θ determines the total number of turns (2π for simple closed curves) [3]. The coefficients a_k, b_k are called Fourier descriptors.

However, such approach has serious drawbacks. First, the curve shape described by a single frequency component

$$\theta_k(s) = s\Theta + a_k \cos(ks\Theta) \qquad (A2)$$

exhibits a wildly varying shape as the coefficient a_k increases (Fig A1, top), thus failing to represent a unique, consistent shape feature. Secondly, in the angle profile representation, linear addition of the frequency components does not properly combine their shape features, but tends to produce rather deformed shapes (Fig A1, bottom). Therefore, decomposing an angle profile into the frequency components via Fourier Transform does not properly translate to a meaningful analysis of the curve's shape into basic shape features.
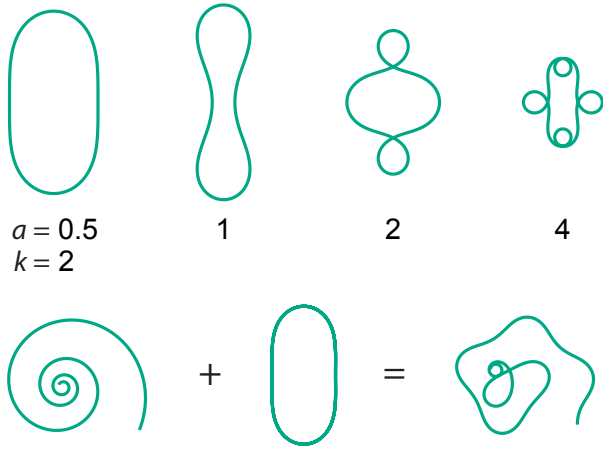
FIG. A1: Problems of angle profile representation. Top: A single frequency component shape described by eq (A2). Bottom: Addition operation in angle profile representation.

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